

Exact self-accelerating cosmologies in the ghost-free massive gravity – the detailed derivation

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We present the detailed derivation of the recently announced most general cosmological solution with homogeneous and isotropic metric in the dRGT ghost-free massive gravity theory. We use the standard parametrization of the theory in terms of the matrix square root, and then show how the same results are recovered within the tetrad formulation. The solution obtained includes the matter source, it exists for generic values of the theory parameters, and it describes a universe that can be spatially open, closed, or flat, and that shows the late time acceleration due to the effective cosmological term mimicked by the graviton mass. The Stückelberg fields are inhomogeneous, which could generate inhomogeneous perturbations of the homogeneous and isotropic background, although this effect should be suppressed by the smallness of the graviton mass.

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I. INTRODUCTION

Considering theories with massive gravitons [1] is motivated by the observation of the current acceleration of our universe [2], since the graviton mass can induce an effective cosmological term. Although such theories typically exhibit unphysical features, as for example the Boulware-Deser ghost [3], the recent discovery of the special massive gravity [4] and its bigravity generalization [5] which are ghost-free [6] suggests that such theories can indeed be good candidates for interpreting the observational data. This motivates studying cosmological solutions with massive gravitons.

The first self-accelerating cosmologies in the ghost-free massive gravity were obtained without matter, they describe the pure de Sitter universe [7]. The matter source was then included for special values of the theory parameters [8]. A surprising feature of these solutions is that their physical and fiducial metrics do not share the same Killing symmetries – although the physical metric is of the Freedman-Robertson-Walker (FRW) type, the fiducial metric is inhomogeneous. This creates a technical difficulty, since the equations for the Stückelberg fields reduce to a non-linear PDE.

One more similar solution was found in Ref.[9], where it was argued that, even though the background geometry is homogeneous and isotropic, the inhomogeneous structure of the Stückelberg fields should be visible when the background is perturbed. However, this effect should be strongly suppressed by the smallness of the graviton mass, so that in small compared to the graviton Compton wavelength regions the deviations from the standard FRW cosmology should be small. These properties seem to be generic for massive gravity cosmologies, since, as was noticed in [9], the ‘genuinely’ homogeneous and isotropic solutions for which both metrics would be FRW do not exist in the theory, at least when the metrics are spatially flat. Even though such solutions were later found in the spatially open case, they are less interesting physically, and show in addition a nonlinear instability [10].

It seems therefore that physical cosmologies with massive gravitons should be described by solutions with a homogeneous and isotropic physical metric, but with an inhomogeneous or anisotropic (or both) fiducial metric (unless the latter is chosen to be non-flat [11]). Such solutions were constructed also in the bigravity context [12], when both metrics are dynamical but are not simultaneously diagonal and do not share the same symmetries. However, in all cases the solutions were obtained only for constrained and not generic

values of the theory parameters. Very recently, a solution for generic parameter values was announced in [15], but without determining the Stückelberg fields, so that the most difficult part of the problem was actually skipped. Finally, the complete solution was obtained in [16], both within the bigravity and massive gravity. In what follows we shall present a detailed derivation of this result.

For pedagogical reasons, we shall restrict our discussion below only to the massive gravity case, since in the bigravity the procedure is essentially the same but the formulas are more complicated. The analysis of [16] was carried out within the tetrad formalism, but we shall employ below the standard parametrization in terms of the matrix square root used by most authors. At the same time, we shall show how the same results are recovered within the tetrad formulation. We shall try to be maximally explicit. For example, it turns out that it is not easy to find in the literature the explicit form of the field equations in the theory, and most authors prefer to put a symmetry ansatz to the action and then vary. We shall therefore show how to vary the action in the general case. We shall also show that the tetrad formulation is equivalent to the standard one.

We make the ansatz for which the physical metric is FRW but the fiducial metric is only spherically symmetric and not diagonal. We then calculate the matrix square root and show that the resulting equations are such that the Stückelberg scalars effectively decouple, and the metric satisfies Einstein equations with an effective cosmological term plus the matter source. However, such a decoupling is only possible if the scalars satisfy a consistency condition expressed by a non-linear PDE. Fortunately, the latter can be solved exactly. As a result, we obtain the most general cosmological solution for which the physical metric is homogeneous and isotropic but the Stückelberg fields are inhomogeneous. The solution includes the matter source, it exists for generic parameter values, and it describes a FRW universe that can be spatially flat, open or closed, and which shows the late-time acceleration due to the effective cosmological term mimicked by the graviton mass.

The rest of the paper is organized as follows. In Sections II, III we introduce the ghost-free massive gravity and derive its equations of motion. The symmetry reduction is described in Section IV. The solution for the metric is presented in Section V, while the consistency condition for its existence is analyzed in Section VI. In Section VII all results are rederived again within the tetrad formulation. Finally, in the Appendix we describe all possible solutions for which both metrics are homogeneous and isotropic.

II. THE GHOST-FREE MASSIVE GRAVITY

The ghost-free massive gravity theory of de Rham, Gabadadze, and Tolley (dRGT) [4] is defined on a four-dimensional spacetime manifold equipped with the metric $g_{\mu\nu}$ and carrying four scalar fields X^A (Stückelberg scalars) which parameterize the fiducial metric $f_{\mu\nu} = \eta_{AB}\partial_\mu X^A\partial_\nu X^B$ with $\eta_{AB} = \text{diag}[1, -1, -1, -1]$. The action is

$$S = \frac{1}{8\pi G} \int \left(-\frac{1}{2} R + m^2 \mathcal{U} \right) \sqrt{-g} d^4x + S_m, \quad (2.1)$$

where m is the graviton mass and S_m describes the ordinary matter (as for example perfect fluid) that interacts with $g_{\mu\nu}$ in the usual way. To define \mathcal{U} , the interaction between $g_{\mu\nu}$ and $f_{\mu\nu}$, the key element is the tensor

$$\gamma^\mu{}_\nu = \sqrt{g^{\mu\alpha} f_{\alpha\nu}},$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. Here the square root is understood in the sense that

$$(\gamma^2)^\mu{}_\nu \equiv \gamma^\mu{}_\alpha \gamma^\alpha{}_\nu = g^{\mu\alpha} f_{\alpha\nu}, \quad (2.2)$$

or, using the hat to denote matrices,

$$\hat{\gamma}^2 = \hat{g}^{-1} \hat{f}. \quad (2.3)$$

Introducing $\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \gamma^\mu{}_\nu$ with traces $[\mathcal{K}] \equiv \text{tr}(\hat{\mathcal{K}}) = \mathcal{K}^\mu{}_\mu$ and $[\mathcal{K}^n] \equiv \text{tr}(\hat{\mathcal{K}}^n) = (\mathcal{K}^n)^\mu{}_\mu$, the interaction is

$$\mathcal{U} = \mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4, \quad (2.4)$$

where α_3, α_4 are parameters, and

$$\begin{aligned} \mathcal{U}_2 &= \frac{1}{2!} ([\mathcal{K}]^2 - [\mathcal{K}^2]), \\ \mathcal{U}_3 &= \frac{1}{3!} ([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + [\mathcal{K}^3]), \\ \mathcal{U}_4 &= \frac{1}{4!} ([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 8[\mathcal{K}^3][\mathcal{K}] - 6[\mathcal{K}^4]). \end{aligned} \quad (2.5)$$

Equivalently, with λ_A being eigenvalues of $\hat{\mathcal{K}}$,

$$\mathcal{U}_2 = \sum_{A<B} \lambda_A \lambda_B, \quad \mathcal{U}_3 = \sum_{A<B<C} \lambda_A \lambda_B \lambda_C, \quad \mathcal{U}_4 = \lambda_1 \lambda_2 \lambda_3 \lambda_4 = \det(\hat{\mathcal{K}}), \quad (2.6)$$

One more equivalent representation is [17]

$$\begin{aligned}\mathcal{U}_2 &= -\frac{1}{2!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\rho\sigma} \mathcal{K}_\alpha^\mu \mathcal{K}_\beta^\nu, \\ \mathcal{U}_3 &= -\frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} \mathcal{K}_\alpha^\mu \mathcal{K}_\beta^\nu \mathcal{K}_\gamma^\rho, \\ \mathcal{U}_4 &= -\frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \mathcal{K}_\alpha^\mu \mathcal{K}_\beta^\nu \mathcal{K}_\gamma^\rho \mathcal{K}_\delta^\sigma.\end{aligned}\tag{2.7}$$

(one has $\epsilon^{0123} = -\epsilon_{0123} = 1$).

III. FIELD EQUATIONS

Some care is needed when varying the action with respect to the metric. Let us first vary the constraint (2.2),

$$\delta\hat{\gamma}\hat{\gamma} + \hat{\gamma}\delta\hat{\gamma} = \delta\hat{g}^{-1}\hat{f},\tag{3.1}$$

which gives

$$\delta\hat{\gamma} + \hat{\gamma}\delta\hat{\gamma}\hat{\gamma}^{-1} = \delta\hat{g}^{-1}\hat{f}\hat{\gamma}^{-1}.\tag{3.2}$$

This cannot be resolved with respect to $\delta\hat{\gamma}$, because $\hat{\gamma}$ and $\delta\hat{\gamma}$ do not commute. However, only the matrix traces enter the action. Taking the trace of (3.2) gives

$$\text{tr}(\delta\hat{\gamma}) + \text{tr}(\hat{\gamma}\delta\hat{\gamma}\hat{\gamma}^{-1}) = 2\text{tr}(\delta\hat{\gamma}) = \text{tr}(\delta\hat{g}^{-1}\hat{f}\hat{\gamma}^{-1}),\tag{3.3}$$

and noting that $\hat{f} = \hat{g}\hat{g}^{-1}\hat{f} = \hat{g}\hat{\gamma}^2$, one obtains

$$\delta[\mathcal{K}] = -\text{tr}(\delta\hat{\gamma}) = -\frac{1}{2} \text{tr}(\delta\hat{g}^{-1}\hat{g}\hat{\gamma}).\tag{3.4}$$

Next, one has

$$\delta[\mathcal{K}^{n+1}] = \delta \text{tr}(\hat{\mathcal{K}}^{n+1}) = (n+1)\text{tr}(\delta\hat{\mathcal{K}}\hat{\mathcal{K}}^n) = -(n+1)\text{tr}(\delta\hat{\gamma}\hat{\mathcal{K}}^n),\tag{3.5}$$

where the cyclic property of the trace is used. Now, multiplying (3.2) by $\hat{\mathcal{K}}^n$ from the right, taking the trace and using the fact that $\hat{\gamma}$ and $\hat{\mathcal{K}}$ commute, one obtains

$$2\text{tr}(\delta\hat{\gamma}\hat{\mathcal{K}}^n) = \text{tr}(\delta\hat{g}^{-1}\hat{f}\hat{\gamma}^{-1}\hat{\mathcal{K}}^n),\tag{3.6}$$

which finally gives

$$\delta[\mathcal{K}^{n+1}] = -\frac{n+1}{2} \text{tr}(\delta\hat{g}^{-1}\hat{g}\hat{\gamma}\hat{\mathcal{K}}^n) = -\frac{n+1}{2} \delta g^{\mu\nu} g_{\nu\beta} \gamma^\beta_\sigma (\mathcal{K}^n)^\sigma_\mu.\tag{3.7}$$

It turns out that the matrix $\hat{g}\hat{\gamma}\hat{\mathcal{K}}^n$ is actually symmetric [13]. To see this, one uses the relation (m is a non-negative integer)

$$\hat{g} \left(\sqrt{\hat{g}^{-1}\hat{f}} \right)^m \hat{g}^{-1} = \left(\sqrt{\hat{f}\hat{g}^{-1}} \right)^m, \quad (3.8)$$

which follows from the fact that squaring it gives identity. This implies that

$$\hat{g} \left(\sqrt{\hat{g}^{-1}\hat{f}} \right)^m = \left(\sqrt{\hat{f}\hat{g}^{-1}} \right)^m \hat{g} \quad (3.9)$$

and hence

$$\left(\hat{g} \left(\sqrt{\hat{g}^{-1}\hat{f}} \right)^m \right)^{\text{tr}} = \left(\left(\sqrt{\hat{f}\hat{g}^{-1}} \right)^m \hat{g} \right)^{\text{tr}} = (\hat{g})^{\text{tr}} \left(\sqrt{(\hat{g}^{-1})^{\text{tr}}(\hat{f})^{\text{tr}}} \right)^m = \hat{g} \left(\sqrt{\hat{g}^{-1}\hat{f}} \right)^m, \quad (3.10)$$

so that $\hat{g}\hat{\gamma}^m$ is a symmetric matrix. Since $\hat{g}\hat{\gamma}\hat{\mathcal{K}}^n$ can be represented as the sum of terms of the form $\hat{g}\hat{\gamma}^m$ with different m , it is also symmetric. This finally gives

$$\frac{\delta[\mathcal{K}^{n+1}]}{\delta g^{\mu\nu}} = -\frac{n+1}{2} g_{\nu\beta} \gamma^\beta_\sigma (\mathcal{K}^n)^\sigma_\mu, \quad (3.11)$$

and the expression on the right here is symmetric with respect to $\mu \leftrightarrow \nu$.

It is now straightforward to vary the action. This gives the Einstein equations

$$G_{\mu\nu} = m^2 T_{\mu\nu} + 8\pi G T_{\mu\nu}^{(\text{m})}, \quad (3.12)$$

where $T_{\mu\nu}^{(\text{m})}$ is the matter energy-momentum tensor obtained by varying S_{m} , while

$$T_{\mu\nu} = 2 \frac{\delta \mathcal{U}}{\delta g^{\mu\nu}} - \mathcal{U} g_{\mu\nu}. \quad (3.13)$$

Using \mathcal{U} in (2.4),(2.5) and applying (3.11) gives

$$\begin{aligned} T_{\mu\nu} &= \gamma_{\mu\alpha} \{ \mathcal{K}_\nu^\alpha - [\mathcal{K}] \delta_\nu^\alpha \} \\ &\quad - \alpha_3 \gamma_{\mu\alpha} \{ \mathcal{U}_2 \delta_\nu^\alpha - [\mathcal{K}] \mathcal{K}_\nu^\alpha + (\mathcal{K}^2)_\nu^\alpha \} \\ &\quad - \alpha_4 \gamma_{\mu\alpha} \{ \mathcal{U}_3 \delta_\nu^\alpha - \mathcal{U}_2 \mathcal{K}_\nu^\alpha + [\mathcal{K}] (\mathcal{K}^2)_\nu^\alpha - (\mathcal{K}^3)_\nu^\alpha \} \\ &\quad - \mathcal{U} g_{\mu\nu}, \end{aligned} \quad (3.14)$$

where $\gamma_{\mu\alpha} = g_{\mu\sigma} \gamma^\sigma_\alpha$. The above considerations guarantee that $T_{\mu\nu} = T_{(\mu\nu)}$. It is also worth mentioning the equivalent representation,

$$T_{\mu\nu} = \gamma_{\mu\alpha} \{ \mathcal{K}_\nu^\alpha - [\mathcal{K}] \delta_\nu^\alpha + \alpha_3 \Xi_\nu^\alpha + \alpha_4 \Omega_\nu^\alpha \} - \mathcal{U} g_{\mu\nu}, \quad (3.15)$$

where

$$\begin{aligned}\Xi_\nu^\alpha &= \frac{1}{2!} \epsilon_{\nu\mu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} \mathcal{K}_\beta^\mu \mathcal{K}_\gamma^\rho, \\ \Omega_\nu^\alpha &= \frac{1}{3!} \epsilon_{\nu\mu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \mathcal{K}_\beta^\mu \mathcal{K}_\gamma^\rho \mathcal{K}_\delta^\sigma.\end{aligned}\tag{3.16}$$

Since the matter energy-momentum tensor is conserved due to the diffeomorphism-invariance of the matter action S_m ,

$$\nabla^\mu T_{\mu\nu}^{(m)} = 0,\tag{3.17}$$

the Bianchi identities for the Einstein equations imply the conservation condition,

$$\nabla^\mu T_{\mu\nu} = 0,\tag{3.18}$$

which can be viewed as equations for the Stückelberg fields.

IV. SYMMETRY REDUCTION AND TAKING THE SQUARE ROOT

Let us now choose spherical coordinates $x^\mu = (t, r, \vartheta, \varphi)$ and assume the physical metric to be homogeneous and isotropic,

$$ds_g^2 = \mathbf{a}(t)^2 dt^2 - \mathbf{a}^2(t) dr^2 - R^2(t, r)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),\tag{4.1}$$

with $R(t, r) = \mathbf{a}(t)f_k(r)$. Here $f_k = \{r, \sin(r), \sinh(r)\}$ for $k = 0, 1, -1$, which corresponds, respectively, to spatially flat, closed, or open FRW universe. The metric is invariant under spatial rotations and translations. We also assume the matter to be a homogeneous and isotropic perfect fluid, so that

$$8\pi G T^{(m)\rho}_\lambda = \text{diag}[\rho(t), -P(t), -P(t), -P(t)].\tag{4.2}$$

The conservation condition (3.17) then reduces to

$$\dot{\rho} + 3\frac{\dot{\mathbf{a}}}{\mathbf{a}}(\rho + P) = 0.\tag{4.3}$$

As for the fiducial metric, we assume it to be only spherically symmetric, but not necessarily homogeneous,

$$\begin{aligned}ds_f^2 &= dT^2(t, r) - dU^2(t, r) - U^2(t, r)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) = \\ &= (\dot{T}^2 - \dot{U}^2)dt^2 + 2(\dot{T}T' - \dot{U}U')dt dr + (T'^2 - U'^2)dr^2 - U^2(t, r)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2).\end{aligned}\tag{4.4}$$

Therefore, the two metrics do not have the same Killing symmetries.

The above expressions imply that

$$(\gamma^2)^\mu{}_\nu = g^{\mu\sigma} f_{\sigma\nu} = \begin{pmatrix} A & C & 0 & 0 \\ -C & B & 0 & 0 \\ 0 & 0 & U^2/R^2 & 0 \\ 0 & 0 & 0 & U^2/R^2 \end{pmatrix}, \quad (4.5)$$

where

$$A = \frac{\dot{T}^2 - \dot{U}^2}{\mathbf{a}^2}, \quad C = \frac{\dot{T}T' - \dot{U}U'}{\mathbf{a}^2}, \quad B = \frac{U'^2 - T'^2}{\mathbf{a}^2}. \quad (4.6)$$

To take the square root of this matrix, one makes the ansatz

$$\gamma^\mu{}_\nu = \begin{pmatrix} a & c & 0 & 0 \\ -c & b & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}. \quad (4.7)$$

Inserting this to (4.5) gives algebraic equations

$$\begin{aligned} a^2 - c^2 &= A, \\ c(a + b) &= C, \\ b^2 - c^2 &= B, \\ u^2 &= U^2/R^2, \end{aligned} \quad (4.8)$$

whose solution is

$$a = \frac{A + \Delta}{Y}, \quad b = \frac{B + \Delta}{Y}, \quad c = \frac{C}{Y}, \quad u = \frac{U}{R}, \quad (4.9)$$

with

$$Y = \sqrt{A + B + 2\Delta}, \quad \Delta = \sqrt{AB + C^2}. \quad (4.10)$$

V. SOLUTION FOR THE METRIC

Since $\gamma^\mu{}_\nu$ is known, we can compute the energy-momentum tensor in (3.14). It is convenient to lift one index and consider $T^\mu{}_\nu = g^{\mu\alpha} T_{\alpha\nu}$. To begin with, the eigenvalues of $K^\mu{}_\nu = \delta^\mu{}_\nu - \gamma^\mu{}_\nu$ are

$$\lambda_{1,2} = \frac{2 - a - b \pm \sqrt{(a - b)^2 - 4c^2}}{2}, \quad \lambda_3 = \lambda_4 = u, \quad (5.1)$$

which can be complex-valued. However, the symmetric polynomials in (2.6) are always real,

$$\begin{aligned}\mathcal{U}_2 &= u(u + 2a + 2b - 6) + c^2 + ab - 3a - 3b + 6, \\ \mathcal{U}_3 &= \alpha_3(1 - u)[(a + b - 2)u + 2c^2 + 2ab - 3a - 3b + 4], \\ \mathcal{U}_4 &= \alpha_4(1 - u)^2(c^2 + ab - a - b + 1).\end{aligned}\tag{5.2}$$

Inserting this and (4.7) into (3.14) gives the following non-zero components:

$$\begin{aligned}T_0^0 &= u(6 - 2b - u) + 3b - 6 + \alpha_3(u - 1)(bu - 2u - 3b + 4) + \alpha_4(u - 1)^2(b - 1), \\ T_r^r &= u(6 - 2a - u) + 3a - 6 + \alpha_3(u - 1)(au - 2u - 3a + 4) + \alpha_4(u - 1)^2(a - 1), \\ T_r^0 &= -c[(\alpha_3 + \alpha_4)u^2 - 2(\alpha_4 + 2\alpha_3 + 1)u + 3 + 3\alpha_3 + \alpha_4], \\ T_\vartheta^\vartheta &= T_\varphi^\varphi = u(3 - a - b) - 6 - ab - c^2 + 3a + 3b + \alpha_4(u - 1)(c^2 + ab - a - b + 1) \\ &\quad + \alpha_3[u(c^2 + ab + 3 - 2a - 2b) - 4 - 2ab + 3b - 2c^2 + 3a].\end{aligned}\tag{5.3}$$

It turns out that

$$c(T_0^0 - T_r^r) = (a - b)T_r^0.\tag{5.4}$$

The field equations $G_\nu^\mu = m^2 T_\nu^\mu + 8\pi G T^{(m)\mu}_\nu$ imply that T_ν^μ should be diagonal, since both G_ν^μ and $T^{(m)\mu}_\nu$ are diagonal. Therefore, one should have

$$T_r^0 = 0,\tag{5.5}$$

and so $T_0^0 = T_r^r$ if $c \neq 0$. Now, T_r^0 in (5.3) will vanish if either $c = 0$, or if the expression between the brackets vanishes. The $c = 0$ case will be considered in the Appendix. If $c \neq 0$, then T_r^0 in (5.3) will vanish if

$$P_2(u) \equiv (\alpha_3 + \alpha_4)u^2 - 2(\alpha_4 + 2\alpha_3 + 1)u + 3 + 3\alpha_3 + \alpha_4 = 0,\tag{5.6}$$

which requires that u is constant,

$$u = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3^2 + \alpha_3 - \alpha_4}}{\alpha_3 + \alpha_4},\tag{5.7}$$

which will be real if $\alpha_4 \leq \alpha_3^2 + \alpha_3 + 1$. Inserting this to (5.3) we find that the two components

$$T_0^0 = T_r^r = (1 - u)(u + u\alpha_3 - \alpha_3 - 3)\tag{5.8}$$

are also constant. The conditions $\nabla_\mu T_\nu^\mu = 0$ reduce in this case to

$$\nabla_\mu T_0^\mu = 2\frac{\dot{\mathbf{a}}}{\mathbf{a}}(T_0^0 - T_\vartheta^\vartheta) = 0,\tag{5.9}$$

which requires that $T_0^0 = T_{\vartheta}^{\vartheta}$. Now, using the above formulas we find that

$$T_0^0 - T_{\vartheta}^{\vartheta} = \frac{u + u\alpha_3 - 2 - \alpha_3}{1 - u} [(u - a)(u - b) + c^2], \quad (5.10)$$

and for this to be zero either the first or the second factor on the right must vanish. Let us require (the other option will be discussed in the next Section) that

$$(u - a)(u - b) + c^2 = 0. \quad (5.11)$$

This condition guarantees that T_{ν}^{μ} is conserved, so that this is the equation for the Stückelberg fields. Assuming that this condition is fulfilled, T_{ν}^{μ} becomes proportional to the unit tensor and the field equations reduce to

$$G_{\lambda}^{\rho} = \Lambda \delta_{\lambda}^{\rho} + 8\pi G T^{(m)}_{\lambda}{}^{\rho} \quad (5.12)$$

with

$$\Lambda = m^2(1 - u)(u + u\alpha_3 - \alpha_3 - 3). \quad (5.13)$$

The functions a, b, c effectively drop out and only the constant u remains. As a result, the effect of the graviton mass is the same as that of a constant cosmological term. Einstein equations (5.12) further reduce to the Friedman equation

$$3 \frac{\dot{\mathbf{a}}^2 + \mathbf{k}\mathbf{a}^2}{\mathbf{a}^4} = \Lambda + \rho, \quad (5.14)$$

where $\rho(\mathbf{a})$ is determined by the conservation condition (4.3). This equation describes a universe filled with matter and containing a cosmological term mimicked by the graviton mass. At early times the matter density ρ dominates, but in the long run the cosmological term wins, leading to the self-acceleration.

VI. SOLUTION FOR THE STÜCKELBERG FIELDS

The above solution for the metric is only possible if equation (5.11) is satisfied, so that the whole procedure is consistent if only this equation can be fulfilled. Rewriting it as

$$ab + c^2 + u^2 = (a + b)u \quad (6.1)$$

and using expressions (4.9) for a, b, c yields

$$\frac{(A + \Delta)(B + \Delta) + C^2}{Y^2} + u^2 = uY. \quad (6.2)$$

Now, using (4.10) one has

$$\begin{aligned}(A + \Delta)(B + \Delta) + C^2 &= AB + C^2 + (A + B)\Delta + \Delta^2 \\ &= (A + B)\Delta + 2\Delta^2 = \Delta(A + B + 2\Delta) = \Delta Y^2,\end{aligned}\quad (6.3)$$

and so (6.2) becomes

$$\Delta + u^2 = uY. \quad (6.4)$$

Next, using (4.6), one finds

$$AB + C^2 = \frac{(\dot{T}U' - \dot{U}T')^2}{\mathbf{a}^4}, \quad (6.5)$$

and therefore

$$\Delta = \sqrt{AB + C^2} = \frac{\dot{T}U' - \dot{U}T'}{\mathbf{a}^2}, \quad (6.6)$$

while

$$Y = \sqrt{A + B + 2\Delta} = \frac{1}{\mathbf{a}} \sqrt{(\dot{T} + U')^2 - (T' + \dot{U})^2}, \quad (6.7)$$

so that (6.4) reduces to

$$\dot{T}U' - \dot{U}T' + u^2\mathbf{a}^2 = u\mathbf{a}\sqrt{(\dot{T} + U' + T' + \dot{U})(\dot{T} + U' - T' - \dot{U})}. \quad (6.8)$$

Squaring this finally gives

$$(\dot{T}U' - \dot{U}T' + u^2\mathbf{a}^2)^2 - u^2\mathbf{a}^2(\dot{T} + U' + T' + \dot{U})(\dot{T} + U' - T' - \dot{U}) = 0. \quad (6.9)$$

This is a quadratic PDE for $T(t, r)$, with the coefficients $\mathbf{a}(t)$ and $U = uR = u\mathbf{a}(t)f_k(r)$ determined by the solution of the Einstein equation (5.14). Although this equation looks complicated, some of its solutions can be obtained.

Let us first consider the case of spatially flat universe, when $U = u\mathbf{a}(t)r$. One makes the ansatz,

$$T(t, r) = f(t) + \mathcal{C}\mathbf{a}(t)r^2, \quad (6.10)$$

with constant \mathcal{C} . In this case Eq.(6.9) reduces to

$$u^2\dot{\mathbf{a}}^2 - 4\mathcal{C}\dot{\mathbf{a}}\dot{f} + 4\mathcal{C}^2\mathbf{a}^2 = 0, \quad (6.11)$$

which can be resolved with respect to $f(t)$ to give

$$T(t, r) = \mathcal{C} \int^t \frac{\mathbf{a}^2}{\dot{\mathbf{a}}} dt + \left(\frac{u^2}{4\mathcal{C}} + \mathcal{C}r^2 \right) \mathbf{a}. \quad (6.12)$$

This solution agrees with the one obtained in [9] for $\alpha_3 = \alpha_4 = 0$, when $u = 3/2$.

When the universe is spatially closed and $U = u\mathbf{a}(t) \sin(r)$, one assumes

$$T(t, r) = f(t) + \mathcal{C}\mathbf{a}(t) \cos(r), \quad (6.13)$$

which reduces Eq.(6.9) to

$$(\mathcal{C}^2 + u^2)(\dot{\mathbf{a}}^2 + \mathbf{a}^2) = \dot{f}^2, \quad (6.14)$$

hence

$$T(t, r) = \pm \int^t \sqrt{(\mathcal{C}^2 + u^2)(\dot{\mathbf{a}}^2 + \mathbf{a}^2)} dt + \mathcal{C}\mathbf{a} \cos(r). \quad (6.15)$$

When the universe is open and $U = u\mathbf{a}(t) \sinh(r)$, one makes the ansatz

$$T(t, r) = f(t) + \mathcal{C}\mathbf{a}(t) \cosh(r), \quad (6.16)$$

reducing the problem to

$$(\mathcal{C}^2 - u^2)(\dot{\mathbf{a}}^2 - \mathbf{a}^2) = \dot{f}^2, \quad (6.17)$$

so that

$$T(t, r) = \pm \int^t \sqrt{(\mathcal{C}^2 - u^2)(\dot{\mathbf{a}}^2 - \mathbf{a}^2)} dt + \mathcal{C}\mathbf{a} \cosh(r). \quad (6.18)$$

This completes our construction, since we have determined the metric and the Stückelberg fields for all three types of the universe and for generic α_3, α_4 .

The above solution exists if $\alpha_4 \leq \alpha_3^2 + \alpha_3 + 1$. Let us consider the limit where this inequality is saturated, $\alpha_4 = \alpha_3^2 + \alpha_3 + 1$. This implies that $u + u\alpha_3 - 2 - \alpha_3 = 0$, in which case the conservation condition (5.9) will be fulfilled without imposing the constraint (5.11). This possibility has been first analyzed in [8] (and recently rediscovered in [14]). The obtained above general solution applies in this case too, and the physical metric is determined by the same equations (5.14) as before. However, the function $T(t, r)$ needs not to be now the same as before, since the constraint (5.11) is no longer imposed, so that there is actually no condition for $T(t, r)$. Therefore, $T(t, r)$ remains arbitrary (a particular choice $T = -\int \dot{U} dr$ was made in [8]).

VII. SOLUTION IN THE TETRAD FORMULATION

Let us now see how the same results are obtained within the tetrad formulation used in [16]. This formulation was originally introduced in [18] and then further developed in [19],

it is equivalent to the standard formulation but can sometimes be more efficient [8],[12]. Although originally it was used to argue in favor of a possible presence of the ghost [18], the absence of ghost in the tetrad description was shown in [19].

The basic variables in the tetrad formulation are two tetrads e_A^μ and ω_ν^B which determine the two metrics,

$$g^{\mu\nu} = \eta^{AB} e_A^\mu e_B^\nu, \quad f_{\mu\nu} = \eta_{AB} \omega_\mu^A \omega_\nu^B, \quad (7.1)$$

and also the tensor

$$\tilde{\gamma}^\mu{}_\nu = e_A^\mu \omega_\nu^A. \quad (7.2)$$

Defining $\mathcal{K}_\nu^\mu = \delta_\nu^\mu - \tilde{\gamma}^\mu{}_\nu$, the action is still given by Eq.(2.1), with the interaction

$$\mathcal{U} = \frac{1}{2} ((\mathcal{K}_\mu^\mu)^2 - \mathcal{K}_\nu^\mu \mathcal{K}_\mu^\nu) - \frac{\alpha_3}{3!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} \mathcal{K}_\alpha^\mu \mathcal{K}_\beta^\nu \mathcal{K}_\gamma^\rho - \frac{\alpha_4}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \mathcal{K}_\alpha^\mu \mathcal{K}_\beta^\nu \mathcal{K}_\gamma^\rho \mathcal{K}_\delta^\sigma \quad (7.3)$$

(the notation $c_3 = -\alpha_3$, $c_4 = -\alpha_4$ was used in [16]). Comparing with (2.4),(2.7), this is the same expression as before, up to the replacement $\gamma_\nu^\mu \rightarrow \tilde{\gamma}_\nu^\mu$. Varying the action with respect to e_A^μ is straightforward and gives the Einstein equations $G_\nu^\mu = m^2 T_\nu^\mu + 8\pi G T^{(m)\mu}_\nu$ with

$$T_\nu^\mu = e_A^\mu \frac{\delta \mathcal{U}}{\delta e_A^\nu} - \mathcal{U} \delta_\nu^\mu. \quad (7.4)$$

Explicitly,

$$T_{\mu\nu} = \tilde{\gamma}_{\mu\alpha} \{ \mathcal{K}_\nu^\alpha - [\mathcal{K}] \delta_\nu^\alpha + \alpha_3 \Xi_\nu^\alpha + \alpha_4 \Omega_\nu^\alpha \} - \mathcal{U} g_{\mu\nu}, \quad (7.5)$$

where $\tilde{\gamma}_{\mu\alpha} = g_{\mu\nu} \tilde{\gamma}_\alpha^\nu$ and Ξ_ν^α and Ω_ν^α defined by (3.16). This expression agrees with that in (3.15), up to the replacement $\gamma_\nu^\mu \rightarrow \tilde{\gamma}_\nu^\mu$. Therefore, if we could show that $\tilde{\gamma}_\nu^\mu = \gamma_\nu^\mu$, this would mean that we have the same equations as before.

The equality $\tilde{\gamma}_\nu^\mu = \gamma_\nu^\mu$ can be enforced by the the following condition:

$$\omega_{A\mu} e_B^\mu = \omega_{B\mu} e_A^\mu, \quad (7.6)$$

with $\omega_{C\mu} = \eta_{CA} \omega_\mu^A$, because

$$\tilde{\gamma}_\alpha^\mu \tilde{\gamma}_\nu^\alpha = e_A^\mu \omega_\alpha^A e_B^\alpha \omega_\nu^B = e^{A\mu} \omega_{A\alpha} e_B^\alpha \omega_\nu^B = e^{A\mu} \omega_{B\alpha} e_A^\alpha \omega_\nu^B = g^{\mu\alpha} f_{\alpha\nu}, \quad (7.7)$$

and therefore $\tilde{\gamma}_\nu^\mu$ fulfills the very same equation which defines γ_ν^μ . The condition (7.6) was originally postulated in [18]. However, it turns out it actually follows from the field equations (see also [19]). Indeed, the Einstein equations imply that $T_{\mu\nu}$ is symmetric. The expression in (7.5) will be always symmetric if only the first four terms on the right are separately

symmetric. Therefore, $\tilde{\gamma}_{\mu\nu}$ should be symmetric, and this guarantees that the other three terms are symmetric as well. As a result, the field equations require that

$$g_{\mu\alpha}\tilde{\gamma}^\alpha_\nu = g_{\nu\alpha}\tilde{\gamma}^\alpha_\mu. \quad (7.8)$$

Using the definition of $\tilde{\gamma}^\alpha_\nu$ and also $g_{\mu\nu} = \eta_{AB}e_\mu^A e_\nu^B$ where e_μ^A is the inverse of e_A^μ , these relations assume the form

$$e_\mu^C \omega_{C\nu} = e_\nu^C \omega_{C\mu}, \quad (7.9)$$

multiplying which by $e_A^\mu e_B^\nu$ gives precisely Eq.(7.6). Therefore, the equality $\tilde{\gamma}^\mu_\nu = \gamma^\mu_\nu$ is imposed dynamically, hence we can remove the tilde sign and conclude that the tetrad formulation gives the same theory as before.

To obtain the solution, one makes the ansatz,

$$\begin{aligned} e_0 &= \frac{1}{\mathbf{a}} \frac{\partial}{\partial t}, & e_1 &= \frac{1}{\mathbf{a}} \frac{\partial}{\partial r}, & e_2 &= \frac{1}{R} \frac{\partial}{\partial \vartheta}, & e_3 &= \frac{1}{R \sin \vartheta} \frac{\partial}{\partial \varphi}, \\ \omega^0 &= \mathbf{a}(a dt + c dr), & \omega^1 &= \mathbf{a}(b dr - c dt), & \omega^2 &= u R d\vartheta, & \omega^3 &= u R \sin \vartheta d\varphi, \end{aligned} \quad (7.10)$$

which fulfills (7.6), with $\mathbf{a} = \mathbf{a}(t)$, and where $R = \mathbf{a}(t)f_k(r)$ is the same as in Eq.(4.1). This implies that $g_{\mu\nu}$ is the same as in (4.1). Computing $\gamma^\mu_\nu = e_A^\mu \omega_\nu^A$ then gives the same result as in (4.7), and therefore all analysis of Section V goes through without any changes.

There remains to analyze the consistency condition (5.11). Let us remember that the metric $f_{\mu\nu} = \eta_{AB}\omega_\mu^A \omega_\nu^B$ should be flat. However, this does not mean that 1-forms $\omega^A = \omega_\mu^A dx^\mu$ coincide with differentials of the Stückelberg scalars, because it is still possible to perform local Lorentz rotations, so that $\omega^A = L^A_B dX^B$ where L^A_B is a position-dependent $SO(1,3)$ matrix. Comparing with $f_{\mu\nu}$ in (4.4), it follows that $\omega^2 = U d\vartheta$ and $\omega^3 = U \sin \vartheta d\varphi$, while ω^0 and ω^1 can differ from dT and dU by a local Lorentz boost,

$$\omega^0 = \cosh(\alpha)dT + \sinh(\alpha)dU, \quad \omega^1 = \cosh(\alpha)dU + \sinh(\alpha)dT, \quad (7.11)$$

where α is the boost parameter. Explicitly,

$$\begin{aligned} \mathbf{a}(a dt + c dr) &= \cosh(\alpha)(\dot{T}dt + T'dr) + \sinh(\alpha)(\dot{U}dt + U'dr), \\ \mathbf{a}(b dr - c dt) &= \cosh(\alpha)(\dot{U}dt + U'dr) + \sinh(\alpha)(\dot{T}dt + T'dr). \end{aligned} \quad (7.12)$$

Comparing the coefficients in front of dt, dr gives four conditions, which determine a, b, c, α in terms of T, U . Inserting the resulting a, b, c to (5.11) then gives precisely the same equation for $T(t, r)$ as in (6.8). Therefore, we recover the same results as before.

In the case when $\alpha_4 = \alpha_3^2 + \alpha_3 + 1$ discussed at the end of Section VI, when the constraint (5.11) is not imposed, there is no condition for coefficients a, b, c, α obtained from (7.12), so that the choice of $T(t, r)$ remains arbitrary. Equivalently, one can choose some value of α and then calculate $T(t, r)$ from (7.12). For example, setting $\alpha = 0$ requires that $T' + \dot{U} = 0$ and so $T = -\int \dot{U} dr$ [8].

VIII. CONCLUSION

We presented the most general cosmological solution in the ghost-free massive gravity with flat reference metric for which the physical metric is homogeneous and isotropic but the Stückelberg fields are inhomogeneous. The solution includes the matter source, it exists for generic values of the theory parameters, and its physical metric describes a FRW universe that can be spatially flat, open or closed, and which shows the late-time acceleration due to the effective cosmological term mimicked by the graviton mass.

Even though the physical metric is homogeneous and isotropic, its perturbations are expected to be inhomogeneous, due to the Stückelberg fields [9]. This effect will be proportional to m^2 and so will be small in small enough regions of space. However, it would be still interesting to compute the linear perturbation spectrum (see also [20]). It would also be interesting to see if the above construction could be generalized to describe non-linear anisotropic deformations of the homogeneous and isotropic background.

We have also shown the equivalence between the standard parametrization of the dRGT theory and the tetrad approach (see also [19]).

APPENDIX. SOLUTIONS WITH FRW METRICS.

For the sake of completeness, we review in this Appendix solutions for which both the physical and fiducial metrics are homogeneous and isotropic. Most of them have been previously reported in the literature. They are obtained by setting $c = 0$ in the formulas of Sections IV,V. Such solutions exist only for particular values of the spatial curvature k and/or show a degenerate fiducial metric. Therefore, they are less interesting physically than solutions described in the main text above.

If $c = 0$, then the condition $T_r^0 = 0$ in (5.5) will be fulfilled without imposing the

constraint (5.6), while Eqs.(4.8) will imply that $C = 0$, so that the metric $f_{\mu\nu}$ in (4.4) is diagonal. Eqs.(4.6),(4.8),(4.9),(4.10) then reduce to

$$a^2 = \frac{\dot{T}^2 - \dot{U}^2}{\mathbf{a}^2}, \quad (\text{A.1a})$$

$$b^2 = \frac{U'^2 - T'^2}{\mathbf{a}^2}, \quad (\text{A.1b})$$

$$\dot{T}T' = \dot{U}U', \quad (\text{A.1c})$$

with $U = u\mathbf{a}(t)f_k(r)$. Here a, b, u are functions of t, r . The energy-momentum tensor is defined by Eqs.(5.3) with $c = 0$ and should satisfy the isotropy condition,

$$T_r^r = T_\vartheta^\vartheta, \quad (\text{A.2})$$

the homogeneity condition, $(T_\nu^\mu)' = 0$, and the conservation condition,

$$\dot{T}_0^0 + 3 \frac{\dot{\mathbf{a}}}{\mathbf{a}} (T_0^0 - T_r^r) = 0. \quad (\text{A.3})$$

The Einstein equations for the metric $g_{\mu\nu}$ then reduce to

$$3 \frac{\dot{\mathbf{a}}^2 + k\mathbf{a}^2}{\mathbf{a}^4} = m^2 T_0^0 + \rho. \quad (\text{A.4})$$

Equations (A.1)–(A.4) determine $a, b, u, \mathbf{a}(t)$, and $T(t, r)$.

Let us consider first the isotropy condition (A.2). Using Eqs.(5.3) gives

$$T_r^r - T_\vartheta^\vartheta = (u - b)[(au - 2a - 2u + 3)\alpha_3 + (1 - u - a + au)\alpha_4 + 3 - a - u], \quad (\text{A.5})$$

and since this has to be zero, either the first or second factor on the right should vanish, which we shall call case I and case II, respectively. In the case I one has $u = b$ and the conservation condition (A.3) reduces to

$$(\dot{\beta} - \dot{\mathbf{a}}a)P_2(u) = 0, \quad (\text{A.6})$$

with $\beta = u\mathbf{a}$ and $P_2(u)$ defined in (5.6). Depending on which of the two factors on the left vanishes, there are two subcases to analyze, let us call them Ia and Ib.

Case Ia: $\dot{\beta} - \dot{\mathbf{a}}a = 0$. One has $a = \dot{\beta}/\dot{\mathbf{a}}$ and $b = u = \beta/\mathbf{a}$. Inserting this to (A.1a) and (A.1b) gives

$$T' = \beta \sqrt{f_k'^2 - 1}, \quad \dot{T} = \dot{\beta} \sqrt{\mathbf{a}^2/\dot{\mathbf{a}}^2 + f_k^2}. \quad (\text{A.7})$$

These should fulfill (A.1c) and also the integrability conditions $\partial_{tr}^2 T = \partial_{rt}^2 T$, which is only possible if $\dot{\beta} = 0$. Therefore, the Stückelberg scalars are

$$T(r) = \beta \int dr \sqrt{f_k'^2 - 1}, \quad U(r) = \beta f_k, \quad (\text{A.8})$$

with constant β . The scale factor satisfies (A.4) with

$$T_0^0 = -4\alpha_3 - \alpha_4 - 6 + \frac{3\beta(3\alpha_3 + \alpha_4 + 3)}{\mathbf{a}} - \frac{3\beta^2(1 + \alpha_4 + 2\alpha_3)}{\mathbf{a}^2} + \frac{\beta^3(\alpha_3 + \alpha_4)}{\mathbf{a}^3}, \quad (\text{A.9})$$

and for $k = 0, \pm 1$. This solution was found in [8] (with the notation $c_3 = \alpha_3$, $c_4 = -\alpha_4$). Its stability has been studied in [21]. Although its physical metric is well behaved, the fiducial metric is degenerate, since both T and U do not depend on t . Moreover, T becomes imaginary for $k = 1$.

Case Ib: $P_2(u) = 0$. This gives the same equation for u as in (5.6). Therefore, one has $b = u$ given by (5.7). Eqs.(A.1b),(A.1c) then yield

$$T' = u \mathbf{a} \sqrt{f_k'^2 - 1}, \quad \dot{T} = \frac{u \dot{\mathbf{a}} f_k f_k'}{\sqrt{f_k'^2 - 1}}, \quad (\text{A.10})$$

and the integrability condition $\partial_{tr}^2 T = \partial_{rt}^2 T$ reduces to $\left(\sqrt{f_k'^2 - 1}\right)'' = \sqrt{f_k'^2 - 1}$, which can be fulfilled only for $k = -1$, when $f_k = \sinh(r)$. As a result, the Stückelberg scalars are

$$T = u \mathbf{a}(t) \cosh(r), \quad U = u \mathbf{a}(t) \sinh(r). \quad (\text{A.11})$$

The scale factor fulfills (A.4) with $k = -1$ and with $m^2 T_0^0 = \Lambda$, where Λ is the same as in (5.13). This solution was found in [10]. The fiducial metrics is non-degenerate, but the solution exists only for $k = -1$. Moreover, it shows a non-linear instability [10].

Let us now consider the case II, when the second factor in (A.5) vanishes, and so

$$a = \frac{(1 + \alpha_4 + 2\alpha_3)u - 3 - \alpha_4 - 3\alpha_3}{(\alpha_3 + \alpha_4)u - 1 - \alpha_4 - 2\alpha_3} \equiv a(u). \quad (\text{A.12})$$

The conservation condition (A.3) then reduces to

$$(b - a)(\dot{u} - u_1) P_1(u) = 0, \quad (\text{A.13})$$

where

$$u_1 = -3 \frac{\dot{\mathbf{a}}}{\mathbf{a}} \frac{P_2(u)}{P_1(u)}, \quad P_1(u) = 2(\alpha_3 + \alpha_4)u - 4\alpha_3 - 2\alpha_4 - 2 = \frac{dP_2(u)}{du}. \quad (\text{A.14})$$

Therefore, there are two options, either $a = b$ or $\dot{u} = u_1$ (if $P_1(u) = 0$, then a diverges).

Case IIa: $b = a$. In this case Eqs.(A.1) yield $T(t, r) = U(t, r)$, so that the metric $f_{\mu\nu}$ is degenerate. This also implies that $a = b = 0$, in which case one obtains from (A.12) $u = (3\alpha_3 + \alpha_4 + 3)/(1 + 2\alpha_3 + \alpha_4)$. The scale factor fulfills Eq.(A.4) with

$$T_0^0 = \frac{\alpha_3^2 + 2\alpha_3 - \alpha_4 + 3}{2\alpha_3 + \alpha_4 + 1}. \quad (\text{A.15})$$

This solution has not been reported in the literature.

Case IIb: $\dot{u} = u_1$. Integrating this equation gives the relation between u and \mathbf{a} ,

$$P_2(u) = \mathcal{A}/\mathbf{a}^3, \quad (\text{A.16})$$

where \mathcal{A} is the integration constant, so that $u = u(t)$ and therefore $a = a(t)$. Eqs.(A.1a) and (A.1c) then yield

$$\dot{T} = \sqrt{\alpha^2 + \dot{\beta}^2 f_k^2}, \quad T' = \frac{\beta \dot{\beta} f_k f_k'}{\sqrt{\alpha^2 + \dot{\beta}^2 f_k^2}}, \quad (\text{A.17})$$

with $\alpha(t) = a\mathbf{a}$ and $\beta(t) = u\mathbf{a}$. The integrability condition $\partial_{tr}^2 T = \partial_{rt}^2 T$ reduce to $\dot{\beta} = \mathcal{B}\alpha$, where \mathcal{B} is an integration constant. Using definitions of α, β , this transforms to

$$\dot{\mathbf{a}} = \frac{\mathcal{B}\mathbf{a} a(u) P_1(u)}{u P_1(u) - 3 P_2(u)}, \quad (\text{A.18})$$

while Eq.(A.1b) yields

$$b = \frac{u f_k'}{\sqrt{1 + \mathcal{B}^2 f_k^2}}. \quad (\text{A.19})$$

Since b depends both on t and r , there are three options to consider. The first two correspond to choosing either $k = \mathcal{B} = 0$, or $k = -1$, $\mathcal{B}^2 = 1$, in which cases $b = u(t)$. The last option is to let b depend on r , but to set $P_2(u) = 0$, in which case the coefficient in front of b in T_0^0 vanishes. In each of these three cases, using (A.18) to eliminate $\dot{\mathbf{a}}$ in the Einstein equation (A.4) and taking into account (A.16), one obtains an algebraic relation containing \mathbf{a} and the matter density ρ . As a result, the solution exists only for fine-tuned ρ , which is unlikely to be physically interesting, so that we do not analyze this case any further.

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